Normal Distribution

Normal distribution is the most important continuous probability distribution in the field of statistics. Since in ~~many~~ applications many random variables are normal random variables or they are approximately normal particularly when the population size is large.

A continuous random variable X with two parameters μ and σ having the p.d.f.

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty \quad \text{(1)}$$

is called the normal variate and the distribution defined by eqn (1) is called the normal distribution.

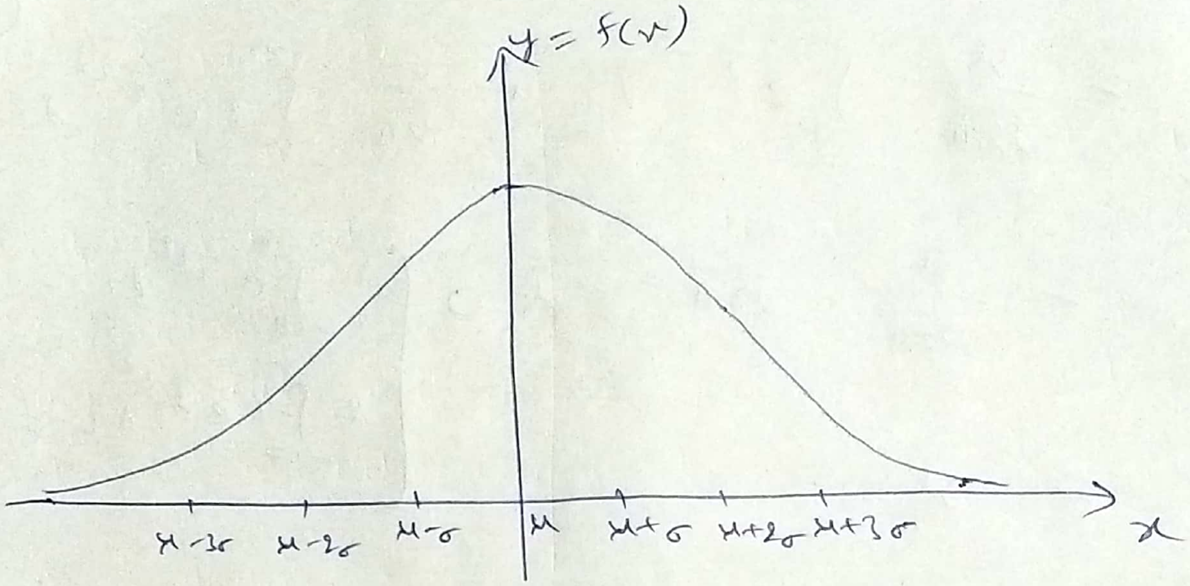
We can easily verify that $f(x)$ defined in eqn (1) is a p.d.f.

(i) $f(x) \geq 0 \quad \forall x$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \qquad \left[\text{use } \Gamma\tfrac{1}{2} = \sqrt{\pi}\right]$

The normal probability curve, $y = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$

is a bell-shaped curve symmetrical about the line $x = \mu$ and attains its maximum value at $x = \mu$.



The normal probability curve describes approximately many phenomena that occur in nature, industry and research.

Q. Find mean and variance of Normal Distribution?

Sol^n

$$E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $z = \frac{x-\mu}{\sigma}$ $\Rightarrow$ $dz = \frac{1}{\sigma} dx$

$\Rightarrow dx = \sigma\, dz$

$$\therefore E(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (\mu + \sigma z)\, e^{-\frac{1}{2}z^2}\, \sigma\, dz$$

③

$$= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} dz$$

$$= \frac{\mu}{\sqrt{2\pi}} \cdot \sqrt{2\pi} + 0 \quad \left[\because \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz = \sqrt{2\pi} \;\&\; \int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} dz = 0 \right]$$

$$= \mu$$

Now $V(x) = E(x-\mu)^2$.

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

let $z = \frac{x-\mu}{\sigma} \Rightarrow x = \sigma z + \mu$

$\Rightarrow dx = \sigma\, dz$

$$\therefore V(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (\sigma z)^2 e^{-\frac{z^2}{2}} \cdot \sigma\, dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2} dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \left[2 \int_{0}^{\infty} z^2 e^{-z^2/2} dz \right] \quad \left\{ \because z^2 e^{-z^2/2} \text{ is an even function of } z \right.$$

(4)

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^\infty \underbrace{z}_{I} \cdot \underbrace{z e^{-z^2/2}}_{II} \, dz.$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \left[z \int_0^\infty z e^{-z^2/2} \, dz - \int_0^\infty \left\{ 1 \cdot \int_0^\infty z e^{-z^2/2} \, dz \right\} dz \right]$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \left[\left(-z e^{-z^2/2} \right]_0^\infty + \int_0^\infty e^{-z^2/2} \, dz \right]$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \left[0 + \sqrt{\frac{\pi}{2}} \right]$$

$$= \sigma^2$$

$\therefore V(x) = \sigma^2$

Hence the standard deviation is

$$S.D. = \sqrt{V(x)} = \sigma.$$

Q. Find the median and mode of the Normal Distribution.

Sol^n Median of Normal Distribution

The median is that value of x which divides the distribution in two equal parts.

Thus, for x to be median,

$$\int_{-\infty}^{x} f(x)\,dx = \int_{x}^{\infty} f(x)\,dx = \frac{1}{2}.$$

Now

$$\int_{-\infty}^{x} f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\mu} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \frac{1}{\sqrt{2\pi}\,\sigma} \int_{\mu}^{x} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

——— 1

Now, consider $I = \int_{-\infty}^{\mu} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$

Let $\frac{x-\mu}{\sigma} = z \Rightarrow dx = \sigma\,dz$

$$\therefore I = \int_{-\infty}^{0} e^{-z^2/2} \cdot \sigma\,dz$$

$$= \sigma \int_{0}^{\infty} e^{-z^2/2}\,dz = \sigma \cdot \sqrt{\frac{\pi}{2}}.$$

Using it in eqn 1, we have.

$$\frac{1}{\sqrt{2\pi}\,\sigma} \cdot \sigma\sqrt{\pi/2} + \frac{1}{\sqrt{2\pi}\,\sigma} \int_{\mu}^{x} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}.$$

$$\Rightarrow \frac{1}{2} + \frac{1}{\sqrt{2\pi}\,\sigma} \int_{\mu}^{x} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}\,\sigma} \int_{\mu}^{x} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 0$$

$$\Rightarrow x = \mu.$$

$\therefore$ $x = \mu$ is the median of the distribution.

Mode of Normal Distribution

The mode is that value of x for which $f(x)$ is maximum.

Therefore, for mode $f'(x) = 0$ & $f''(x) < 0$.

Now p.d.f. of normal distribution is

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Taking logarithm both sides

$$\ln f(x) = \ln\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right) - \frac{1}{2\sigma^2}(x-\mu)^2$$

$$\Rightarrow \ln f(x) = k - \frac{1}{2\sigma^2}(x-\mu)^2 \quad \text{——— (3)}$$

where $k = \ln\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)$ is a constant

Now Diff. w. r. to x, we have

$$f'(x) = -\frac{1}{\sigma^2}(x-\mu)\, f(x)$$

$$\& \quad f''(x) = \frac{-f(x)}{\sigma^2}\left[1 - \frac{(x-\mu)^2}{\sigma^2}\right]$$

$f'(x) = 0$ gives $x = \mu$

and $\left[f''(x)\right]_{x=\mu} = -\frac{1}{\sigma^2}\left[f(x)\right]_{x=\mu}$

$$= -\frac{1}{\sigma^2} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} < 0$$

Hence $x = \mu$ is the mode of the distribution.

Hence, for a Normal Distribution

Mean = Median = Mode.

Thus, The Normal Distribution is Symmetrical.

Q. Find Moment Generating function for Normal Distribution.

Soln The m.g.f is given by

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $\frac{x-\mu}{\sigma} = z \Rightarrow dx = \sigma dz$

$$\therefore M_x(t) = e^{\mu t} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2 + t\sigma z} dz$$

$$= e^{\mu t + \frac{t^2\sigma^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2} dz$$

Let $u = z - t\sigma \Rightarrow du = dz$

$$\therefore M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du$$

$$\Rightarrow M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi}$$

$$\Rightarrow M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

The m.g.f. about mean μ is

$$M_\mu(t) = E\left[e^{t(x-\mu)}\right] = e^{-\mu t} \cdot E(e^{tx})$$

$$= e^{-\mu t} \cdot M_x(t) = e^{-\mu t} \cdot e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$= e^{\frac{\sigma^2 t^2}{2}}$$

Moments about the Mean

$\because$ The m.g.f. about mean μ is

$$M_\mu(t) = e^{\frac{1}{2}\sigma^2 t^2}$$

$$\Rightarrow M_\mu(t) = 1 + \left(\frac{1}{2}t^2\sigma^2\right) + \frac{1}{2!}\left(\frac{t^2\sigma^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{t^2\sigma^2}{2}\right)^3 + \cdots + \frac{1}{n!}\left(\frac{t^2\sigma^2}{2}\right)^n + \cdots \quad \text{——— (4)}$$

Now we know that, Moments about mean are

$$\mu_r = E\left((X-\mu)^r\right) = \left\{\frac{d^r}{dt^r}\left[M_\mu(t)\right]\right\}_{t=0}$$

from eqn (4),

$$\mu_{2n+1} = \text{Coeff. of } \frac{t^{2n+1}}{(2n+1)!} = 0$$

and

$$\mu_{2n} = \text{Coeff. of } \frac{t^{2n}}{(2n)!} = \frac{\left(\frac{1}{2}\sigma^2\right)^n (2n)!}{n!}$$

$$\Rightarrow \mu_{2n} = 1.3.5.7.\cdots(2n-1)\,\sigma^{2n}$$